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MATHEMATICAL EXPRESSIONS FOR CLOUD DROPLET SIZE DISTRIBUTIONS AND THE PARAMETERIZATION OF EFFECTIVE RADIUS

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Mathematical expressions for cloud droplet size distributions and the parameterization of effective radius

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Abstract. A number of mathematical expressions (e.g., Gamma distribution, lognormal distribution and Weibull distribution) have been proposed to describe cloud droplet size distributions in cloud-related studies such as microphysics parameterization in cloud-resolving models and remote sensing of cloud properties. Similarly, several mathematical expressions have been proposed to parameterize the effective radius as a function of the droplet concentration and the liquid water content. However, there is no consensus in either area as to the expression that best represents ambient clouds. It is shown that the question of the best size distribution expression can be addressed by comparing measured values of effective radius to those derived from the different parameterizations of effective radius because of the unique correspondence between the commonly used expressions for the parameterization of effective radius and those for describing size distributions. Analysis of data collected in marine stratus and stratocumulus clouds during the 1993 North Atlantic Regional Experiment reveals that the Weibull distribution most accurately represents observed size distributions. The performance of the Gamma distribution is close to the Weibull distribution. Parameterizations based on lognormal and Gaussian distributions overestimate and underestimate the effective radius, respectively. It is further illustrated that the errors in the effective radius from some of the parameterization schemes are large enough to cause serious errors in climate modeling and the interpretation of cloud remote sensing. The results of this study further emphasize the necessity of predicting the spectral dispersion of droplet size distributions in addition to the liquid water content and droplet concentration to meet the need of reducing uncertainties in climate models as suggested in our recent study.

1. Introduction

Cloud droplet size distribution is a fundamental property of clouds that is important in almost all cloud-related areas. For examples, correctly specifying the mathematical expression of the droplet size distribution is necessary for modeling the evolution of the size distribution by means of the so-called moment method [*Williams and Loylka*, 1991; *Mitchell et al.*, 1996], for parameterization of cloud microphysics in cloud-resolving modeling[*Walko et al.*, 1995; *Meyers et al.*, 1997; *Feingold et al.*, 1998], and for remote sensing of cloud properties [*Goddard et al.*, 1997; *Dong et al.*, 1997]. A number of mathematical functions (e.g., Gamma distribution, Weibull distribution and lognormal distribution) have been proposed and used to describe droplet size distributions, but there has been no agreement as to which of these expressions is the most appropriate for these purposes.

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The influence of droplet size distributions on radiative properties of clouds is often parameterized as a function of the effective radius r_e [*Hansen and Travis*, 1974; *Slingo*, 1989]. The parameterization of r_e has recently become a subject of active research because it has proven to be a quantity critical for assessing cloud-radiation-climate interactions, the indirect effects of anthropogenic aerosols on global climate change [*Slingo*, 1990; *Schwartz and Slingo*, 1996; *Dandin et al.*, 1997; *Rotstayn*, 1999; *Hu and Stamnes*, 2000], and for the remote sensing of cloud properties [*Li et al.*, 1999]. Similar to the situation in choosing a representation for size distributions, a number of parameterizations for r_e have been developed, but it is not clear which of these parameterizations provides the best representation of r_e.

For the most part, the specification of size distribution expressions and the parameterization of r_e have been addressed separately. The primary objectives of this contribution are: (1) to show the connection between the parameterization of r_e and the mathematical expression assumed to describe the droplet size distribution, (2) to identify the best expression for droplet size distributions and hence the best parameterization of r_e, and, (3) to quantify the differences in r_e estimated using different parameterizations and thereby demonstrate the importance of choosing the expression that most accurately represents droplet size distributions.

2. Relationship between Size Distribution Expressions and r_e Parameterizations

To show the connection between the parameterization of r_e and the mathematical expression used to describe droplet size distributions, we first analyze the existing parameterizations of r_e in the context of the different assumptions that these parameterizations make with respect to droplet size distributions. It is generally agreed that r_e can be parameterized as the cube root of the ratio of the cloud liquid water content (L) to the droplet concentration (N) [*Pontikis and Hicks*, 1992; *Bower et al.*, 1994; *Bower et al.*, 1994; *Martin et al.*, 1994; *Liu and Hallett,* 1997; *Reid et al.*, 1999; *Liu and Daum*, 2000].

$$r_e = \alpha \left(\frac{L}{N} \right)^{1/3}, \tag{1}$$

where r_e is in μm, L in g m^{-3}, N in cm^{-3}, and α the prefactor. Early users of parameterizations of this form assumed fixed values of α which correspond to very narrow droplet size distributions in clouds with weak turbulent entrainment and mixing [*Bower et al.*, 1994; *Bower et al.*, 1994; *Martin et al.*, 1994]. These parameterizations are referred to as monodisperse-like and denoted by ML hereafter. Assuming that droplet size distributions are negligibly skewed, an improved parameterization was formulated which relates α to the spectral dispersion d of the corresponding droplet size distribution. It is hereafter referred to the Gaussian-like and denoted by GL because the Gaussian distribution is a typical form of symmetrical distributions with skewness of 0. Although it is well known in cloud physics that neither monodisperse-like nor Gaussian-like distributions are good representations of droplet size distributions in real clouds, the parameterization schemes for r_e based on such idealizations have been used, and are still in wide use to treat cloud effects in climate models. Recently we showed that the parameterization scheme for r_e based on the Weibull form of size distributions (WB hereafter) is superior to those that assume either Monodisperse-like or Gaussian-like distributions [*Liu and Daum*, 2000].

Similar to the derivation of α for the Weibull distribution [*Liu and Hallett*, 1997], expressions for α as a function of d can be easily derived for the Gamma [*Han et al.*, 1998; GM hereafter] and lognormal [*Gerber*, 1996; LN hereafter] droplet size distributions. Table 1 summarizes the commonly used mathematical expressions for describing droplet size distributions and the corresponding expressions for α as a function of d. It is obvious from this table that the only distinction between these different parameterizations for r_e is the form of the dependency of α on d, which is determined by the functional form that is assumed for droplet size distributions. For this reason we argue that the determination of the best mathematical expression of the droplet size distribution is equivalent to the identification of the best parameterization of r_e, which in turn is equivalent to choosing the expression that best characterizes the dependence of α on d.

Table 1. Expressions for Prefactor α in the "1/3" Power-Law

MO	$n(r) = N\delta(r - r_0)$	$\alpha = 62.04$
MM	*narrow size distribution*	$\alpha = 66.84$
MC	*narrow size distribution*	$\alpha = 70.91$
GL	*Gaussian-like distribution*	$\alpha = 62.04\dfrac{\left(1+3\mathrm{d}^2\right)^{2/3}}{\left(1+\mathrm{d}^2\right)}$
WB	$n(r) = N_0 r^{b-1}\exp\left(-\lambda r^b\right)$	$\alpha(\mathrm{b}) = 64.52\dfrac{\Gamma^{2/3}(3/\mathrm{b})}{\Gamma(2/\mathrm{b})}\mathrm{b}^{1/3},\ \mathrm{d} = \left[\dfrac{2\mathrm{b}\Gamma\left(2/\mathrm{b}\right)}{\Gamma^2\left(1/\mathrm{b}\right)} - 1\right]^{1/2}$
GM	$n(r) = N_0 r^{\mu}\exp\left(-\lambda r\right)$	$\alpha = 62.04\dfrac{\left(1+2\mathrm{d}^2\right)^{2/3}}{\left(1+\mathrm{d}^2\right)^{1/3}}$
LN	$n(r) = \dfrac{N}{\sigma\sqrt{2\pi}}\dfrac{1}{r}\exp\left(-\dfrac{\ln^2\left(r/r_m\right)}{2\sigma^2}\right)$	$\alpha = 62.04\left(1+\mathrm{d}^2\right)$

MO = monondisperse size distributions; MM = *Martin et al.* [1994] for marine clouds; MC = *Martin et al.* [1994] for continental clouds; GL = Gaussian-like distribution; WB = Weibull distribution; GM = Gamma distribution; LN = Lognormal distribution.

3. Identification of Size Distribution Expressions and r_e Parameterizations

The most accurate parameterization of r_e and the best expression of the droplet size distribution can be identified simultaneously by comparing d's and α's computed using the various expressions listed in Table 1 to experimental values of these quantities in a (d, α) diagram. The analysis is based on 10s data collected using a

Forward Scattering Aerosol Spectrometer Probe (FSSP, Particle Measurement Systems Inc, Boulder, Colorado). The measurements were made during the 1993 North Atlantic Regional Experiment (NARE) 1993 in marine stratus and stratocumulus clouds over the North Atlantic Ocean off the southern tip of Nova Scotia in August and September. Details regarding the measurements can be found in *Leaitch et al.* [1996]. Data from 13 flights were used.

The (d, α) plot for the data is shown in Figure 1. For the sake of clarity, data from the various flights were first partitioned into groups according to d's with incremental interval of 0.03 and the data in each group for all 13 flights were then averaged. Each point in the plot (solid dots) represents one of these averages. The behavior of data from the individual flights exhibits a dependency very similar to the one shown in Figure 1. The lines in Figure 1 labeled GL, WB, GM and LN represent the corresponding distribution families. Note that the data points neither cluster around the fixed values of α as used in the ML parameterizations, nor do they follow the dependence of GL or LN expressions. The GL tends to underestimate α, whereas the LN tends to overestimate α. The WB expression appears to most accurately describe the dependence of α on d over the observed range of d, although the differences between the WB and the GM expressions are quite small.

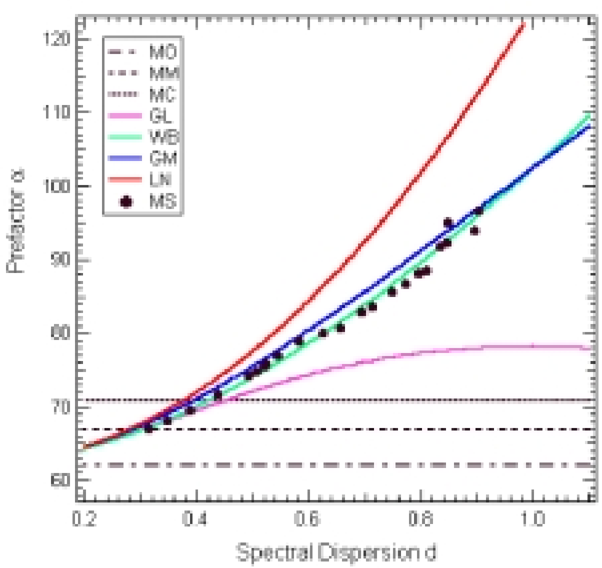

Figure 1. The (d, α) diagram showing the comparison of the measured prefactor α as a function of spectral dispersion d to those calculated from the different parameterizations. The meanings of symbols (MO, MM, MC, GL, WB, GM, LN) are referred to Table 1. The solid dots represent averages derived from the FSSP-measured cloud droplet size distributions.

4. Further Analysis

The above analysis clearly shows differences between the different parameterizations. The question arises as to whether the differences are significant enough to deserve serious consideration in parameterizing r_e. It is expected from Eq. (1) that the differences in α will result in similar differences in estimates for r_e.

Figure 2 displays the differences between r_e measured by the FSSP (r_{em}) and those estimated from the different parameterization schemes as a function of r_{em}. In this figure it is clear that the WB parameterization for r_e yields values that are closest to the measured values of r_e, with errors within 0.3 μm, or ~3%. Next is the GM parameterization with errors within 0.5 μm, or ~5%. The LN parameterization tends to overestimate r_e, and the overestimation increases with r_{em}, up to 3 μm, or ~30%. On the contrary, the GL parameterization tends to underestimate r_e and the underestimation increases with r_{em}, up to 3μm, or ~30%. As shown in Figure 3, the inappropriate treatment of α is the reason for the bias in the corresponding parameterized r_e's shown in Figure 2. The increase of the difference with r_{em} is mainly due to the fact that droplet size distributions broaden toward larger sizes.

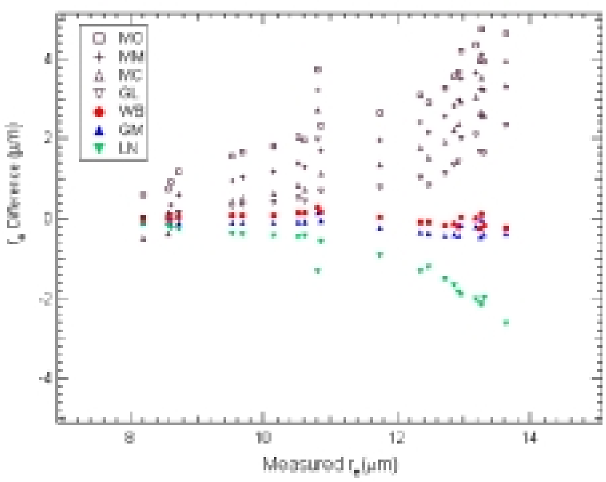

Figure 2. The difference between measured effective radius and those estimated from different parameterizations as a function of the measured effective radius. The meanings of symbols (MO, MM, MC, GL, WB, GM, LN) are referred to Table 1.

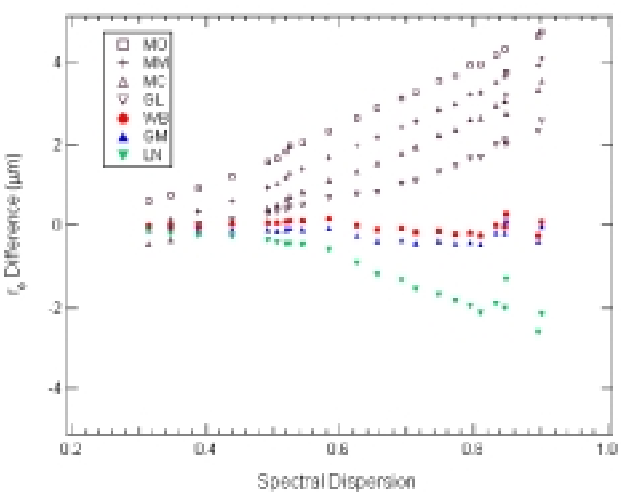

Figure 3. The difference between measured effective radius and those estimated from different parameterizations as a function of the spectral dispersion. The meanings of symbols (MO, MM, MC, GL, WB, GM, LN) are referred to Table 1.

Previous studies have shown that the top-of-atmosphere radiative forcing of doubling the CO_2 concentration could be offset by reducing r_e of low clouds from 10 μm to between 7.9 and 8.6 μm (approximately 15-20%), depending on the climate model used to make the prediction [*Slingo*, 1990]. Changing r_e from 10 μm to 7 μm (approximately 30%) could substantially reduce a recently reported discrepancy between model-predicted and observed cloud absorption [*Li et al.*, 1999]. A more recent study indicated that a 10 % increase in r_e could increase the surface temperature by about 1.6˚C, about the same as predicted to result from the doubling of the CO_2 concentration [*Hu and Stammes*, 2000]. It is evident from Figure 2 and Figure 3, that the differences between r_e estimated from the different parameterizations are large enough to cause serious problems in climate models, if the effects of α (or d) are not treated properly (e.g., see the difference between MO and LN). The issue of accurately specifying d and its effect on the radiative properties of clouds is important especially when cloud inhomogeneity and 3D structure of clouds are considered, because d can vary widely in clouds depending on the position in clouds, the growth time of clouds and the turbulence intensity involved [*Warner*, 1968; *Costa et al.*, 2000]. This issue could be more important when both the radiative balance and the hydrological circle of the Earth is concerned [*Chahine*, 1990].

5. Conclusions

The above analysis suggests that droplet size distributions of the marine stratus and stratocumulus clouds examined are best described by the Weibull distribution family in the context of the parameterization of r_e. *Costa et al*. (2000) also found that the Weibull distribution best fit droplet size distributions observed in both marine and continental cumuli among the tested mathematical expressions (exponential distribution, Gamma distribution, lognormal distribution and Weibull distribution). The success of the Weibull distribution relative to the other commonly used distributions is worth emphasizing, because less attention has been given to it. In fact, the Weibull distribution was introduced to describe droplet size distributions as early as 1940 [*Schumann*, 1940], earlier than the seminal work by *Weibull* [1951]. The lognormal distribution is also widely used to represent turbulent fluctuations, a subject closely related to droplet size distributions. The use of lognormal distribution for this purpose has been recently criticized [*Mandelbrot*, 1997], and it appears that similar criticism may apply to the description of droplet size distributions as well. Recently, a physical explanation justifying the use of the Weibull form of droplet size distributions was proposed by integrating into cloud physics the ideas of statistical mechanics and information theory [*Liu et al.*, 1995; *Liu and Hallett*, 1997; 1998]. The gamma distribution was also shown to represent the ensemble-averaged droplet size distributions from the stochastic theory of condensation [*Khvorostyanov and Curry*, 1999]. However, we are not aware of any physical explanations for the lognormal size distribution.

It is well known that classical condensation theory predicts monodisperse-like or Gaussian-like droplet size distributions which are known to be much narrower and more symmetrical than observed size distributions. The reconciliation of this discrepancy is a long-standing challenge for the cloud physics community. It is interesting to note that the climate model community has been using the parameterizations of r_e that essentially correspond to these narrow, symmetrical size distributions. Lognormal distributions have been uncritically used in remote sensing of cloud properties and microphysics parameterizations in cloud-resolving models as well. This study reveals the importance of choosing the best representation of droplet size distributions to parameterize r_e. It further highlights the necessity of predicting α (or d) in addition to L and N in climate models suggested in our previous study [*Liu and Daum*, 2000].

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References

Bower, K. N. and T. W. Choularton, A parameterization of the effective radius of ice-free clouds for use in global climate models, *Atmos. Res., 27*, 305-339, 1992.

Bower, K. N., T. W. Choularton, J. Latham, J. Nelson, M. B. Baker, and J. Jensen, A parameterization of warm clouds for use in atmospheric general circulation models, *J. Atmos. Sci., 51*, 2722-2732, 1994.

Chahine, M. T., The hydrological cycle and its influence on climate, Nature, 359, 373-379, 1992.

Costa, A. A., C. Oliveira, J. Oliveira, and A. Sampaio, Microphysical observations of warm cumulus clouds in Ceara, Brazil, *Atmos. Res., 54*, 167-199, 2000.

Dandin, P., C. Pontikis, and E. Hicks, Sensitivity of a GCM to changes in the droplet effective radius parameterization, *Geophys. Res. Lett., 24*, 437-440, 1997.

Dong, X., P. T. P. Ackermann, E. E. Clothiaux, P. Pilewskie, and Y. Han, Microphysical and radiative properties of boundary layer stratiform clouds deduced from ground-based measurements. *J. Geophys. Res*., *D102*, 23829-23843, 1997.

Feingold, G., R. L., Walko, B. Stevens, and W. R. Cotton, Simulations of marine stratocumulus using a new microphysical parameterization scheme, *Atmos. Res., 47-48*, 505-528, 1998

Gerber, H, Microphysics of marine stratocumulus clouds with two drizzle modes, *J. Atmos. Sci., 53*, 1649-1662, 1996

Gossard, E. E., J. B. Snider, E. E. Clothiaux, B. Martner, J. S. Gibson, R. A. Kropfli, and A. S. Frish, The potential of 8-mm radars for remote sensing cloud droplet size distributions, *J. Atmos. Oceanic Technol., 14*, 76-87, 1997

Han, Q., W. B. Rossow, J. Chou, R. M. Welch, Global variation of column droplet concentration in low-level clouds, *Geophys. Res. Lett., 25*, 1419-1422, 1998.

Hansen, J. E., and L. D. Travis, Light scattering in planetary atmospheres, *Space Sci. Rev., 16*, 527-610, 1974.

Hu, Y., and S. Stamnes, Climate sensitivity to cloud optical properties, *Tellus, 52B*, 81-93, 2000

Khvorostyanov, V. I., and J. A. Curry, Toward the theory of stochastic condensation in clouds. part II: Analytical solutions of the Gamma-distribution type, *J. Atmos. Sci., 56*, 3997-4013, 1999.

Leaitch, W. R., C. M., Banic, G. A. Isaac, M. D. Couture, P. S. K. Liu, I. Gultepe, S. Liu, L. Kleinman, P. H. Daum, and J. I. MacPherson, Physical and chemical observations in marine stratus during the 1993 North Atlantic Regional Experiment. Factors controlling cloud droplet number concentration, *J. Geophys. Res., D101*, 29,123-29,135, 1996

Li, Z., A. P., Trishchenko, H. W. Barker, G. L. Stephens, and P. Partain, Analyses of Atmospheric Radiation Measurement (ARM) Program's Enhanced Shortwave Experiment (ARESE) multiple data sets for studying cloud absorption, *J. Geophys. Res., D104*, 19,127-19,134, 1999

Liu, Y., and P. H. Daum, Spectral dispersion of cloud droplet size distributions and the parameterization of cloud droplet effective radius, *Geophy. Res. Lett., 27*, 1903-1906, 2000

Liu, Y., and J. Hallett, The "1/3" power-law between effective radius and liquid water content, *Q. J. R. Meteor. Soc., 123*, 1789-1795, 1997.

Liu, Y., and J. Hallett, On size distributions of cloud droplets growing by condensation, A new conceptual model, *J. Atmos. Sci., 55*, 527-536, 1998.

Liu, Y., L. You, W. Yang, and F. Liu, On the size distribution of cloud droplets, *Atmos. Res., 35*, 201-216, 1995.

Mandelbrot, B. B., *Multifractals and 1/f Noise*, 442 pp., Springer, New York, 1999.

Martin, G. M., D. W. Johnson, and A. Spice, The measurement and parameterization of effective radius of droplets in the warm stratocumulus clouds, *J. Atmos., Sci., 51*, 1823-1842, 1994.

Meyers, M. P., R. L. Walko, J. Y. Harrington, W. R. Cotton, New RAMS cloud microphysics parameterization, part 2, The two-moment scheme, *Atmos. Res., 45*, 3-39, 1997

Mitchell, D. L, S. K. Chai, Y. Liu, A. J. Heymsfield, and Y. Dong, Modeling cirrus clouds, part 1, Treatment of bimodal size spectra and case study analysis, *J. Atmos. Sci., 53*, 2952-2966, 1996.

Pontikis, C., and E. M. Hicks, Contributions to the droplet effective radius parameterization, *Geophy. Res. Lett., 19*, 2227-2230, 1992.

Reid, S. J., P. V. Hobbs, A. L. Rangno, and D. A. Hegg, Relationship between cloud droplet effective radius, liquid water content, and droplet concentration for warm clouds in Brazil embedded in biomass smoke, *J. Geophy. Sci., D104*, 6145-6153, 1999.

Rotstayn, L. D., Indirect forcing by anthropogenic aerosols, A global climate model calculations of the effective-radius and cloud-time effects, *J. Geophys. Res., D104*, 9369-9380, 1999.

Schwartz, S. S., and A. Slingo, Enhanced shortwave cloud radiative forcing due to anthropogenic aerosols, in *Clouds, Chemistry and Climate*, edited by P. Crudzen and V. Ramanathan, pp. 191-236, Spinger, Heidelberg, 1996.

Slingo, A., A GCM parameterization for the shortwave radiative properties of water clouds, *J. Atmos. Sci., 46*, 1419 – 1427, 1989

Slingo, A, Sensitivity of the Earth's radiation budget to changes in low clouds, *Nature, 343*, 49-51,1990

Shumann, Theoretical aspects of the size distribution of fog particles, *Q. J. R. Meteoro. Soc., 66*, 195-207, 1940.

Walko, R. L., W. R. Cotton, M. P. Meyers, J. Y. Harrington, New RAMS cloud microphysics parameterization, part 1, the single-moment scheme, *Atmos. Res., 38*, 29-62, 1995

Warner, J., Gust velocities and the droplet spectrum in cumulus clouds, *Proc. Int. Conf. on Cloud Phys.*, pp. 138-140, Toronto, Canada, 1968.

Weibull, W., A statistical distribution function of wide applicability, *J. Appl. Mech., 18*, 293-297, 1951.

Williams, M. M. R., and S. K. Loyalka, *Aerosol Science: Theory and Practive*, Pergamon Press, London, 1991.

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